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# Families of solutions of order nine to the NLS equation with sixteen parameters.

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## Abstract

We construct new deformations of the Peregrine breather ( $P_9$ ) of order 9 with 16 real parameters. With this method, we obtain explicitly new families of quasi-rational solutions to the NLS equation in terms of a product of an exponential depending on  $t$  by a ratio of two polynomials of degree 90 in  $x$  and  $t$ ; when all the parameters are equal to 0, we recover the classical  $P_9$  breather. We construct new patterns of different types of rogue waves as triangular configurations of 45 peaks as well as rings and concentric rings.

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**Keywords :** NLS equation, Peregrine breather, rogue waves.

# 1 Introduction

After the works of the precursors, Zakharov and Shabat in 1972 [1, 2], a considerable number of studies were carried out by a multiplicity of authors. The first expressions of the quasi-rational solutions were given by Peregrine in 1983 [3]. Akhmediev, Eleonski and Kulagin constructed the first higher order analogue of the Peregrine breather [4, 5] in 1986. Akhmediev et al. [6, 7], constructed other families of order 3 and 4, using the Darboux transformations.

It has been found in [10] solutions for the order  $N$  (for determinants of order  $2N$ ) to the NLS equation depending on  $2N - 2$  real parameters.

With this method, we construct explicitly here new quasi rational solutions to the NLS equation for the order 9 depending on sixteen parameters. When all the parameters are equal to zero, we recover the famous (analogue) Peregrine breather  $P_9$ . We obtain new deformations at order 9 with 16 real parameters of the  $P_9$  breather as a ratio of two polynomials of degree 90 in  $x$  and  $t$  by an exponential depending on  $t$ . Because of the length of these expressions, we cannot present them here; we only present patterns depending on the choices of the parameters and make the analysis of the evolution of the solutions.

## 2 Determinant representation of solutions to NLS equation

We recall briefly the results obtained in [11, 12] that we use in the follow-

ing to construct deformations of the  $P_9$  breather, solutions to the NLS equation.

**Theorem 2.1** *The function  $v$  defined by*

$$v(x, t) = \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} e^{(2it - i\varphi)}$$

*is a quasi-rational solution of the NLS equation*

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where

$$\begin{aligned} n_{j1} &= f_{j,1}(x, t, 0), \\ n_{jk} &= \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ n_{jN+1} &= f_{j,N+1}(x, t, 0), \\ n_{jN+k} &= \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{j1} &= g_{j,1}(x, t, 0), \\ d_{jk} &= \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{jN+1} &= g_{j,N+1}(x, t, 0), \\ d_{jN+k} &= \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ 2 \leq k \leq N, 1 \leq j \leq 2N \end{aligned}$$

The functions  $f$  and  $g$  are defined for  $1 \leq k \leq N$  by :

$$\begin{aligned} f_{4j+1,k} &= \gamma_k^{4j-1} \sin A_k, \\ f_{4j+2,k} &= \gamma_k^{4j} \cos A_k, \\ f_{4j+3,k} &= -\gamma_k^{4j+1} \sin A_k, \\ f_{4j+4,k} &= -\gamma_k^{4j+2} \cos A_k, \\ f_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos A_{N+k}, \\ f_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin A_{N+k}, \\ f_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos A_{N+k}, \\ f_{4j+4,k} &= \gamma_k^{2N-4j-5} \sin A_{N+k}, \\ g_{4j+1,k} &= \gamma_k^{4j-1} \sin B_k, \\ g_{4j+2,k} &= \gamma_k^{4j} \cos B_k, \\ g_{4j+3,k} &= -\gamma_k^{4j+1} \sin B_k, \\ g_{4j+4,k} &= -\gamma_k^{4j+2} \cos B_k, \\ g_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos B_{N+k}, \\ g_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin B_{N+k}, \\ g_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos B_{N+k}, \\ g_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin B_{N+k}, \end{aligned} \tag{1}$$

The arguments  $A_\nu$  and  $B_\nu$  of these functions are given for  $1 \leq \nu \leq 2N$  by

$$\begin{aligned} A_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ B_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2. \end{aligned}$$

The terms  $\kappa_\nu$ ,  $\delta_\nu$ ,  $\gamma_\nu$  are defined by  $1 \leq \nu \leq 2N$

$$\begin{aligned} \kappa_j &= 2\sqrt{1 - \lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \\ \gamma_j &= \sqrt{\frac{1 - \lambda_j}{1 + \lambda_j}}, \quad \kappa_{N+j} = \kappa_j, \\ \delta_{N+j} &= -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \\ 1 &\leq j \leq N, \end{aligned} \quad (2)$$

where  $\lambda_j$  are given for  $1 \leq j \leq N$  by :

$$\lambda_j = 1 - 2j^2\epsilon^2, \quad \lambda_{N+j} = -\lambda_j. \quad (3)$$

The terms  $x_{r,\nu}$  ( $r = 3, 1$ ) are defined for  $1 \leq \nu \leq 2N$  by :

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}. \quad (4)$$

The parameters  $e_\nu$  are given by

$$\begin{aligned} e_j &= i \sum_{k=1}^{N-1} \tilde{a}_j \epsilon^{2k+1} j^{2k+1} \\ &\quad - \sum_{k=1}^{N-1} \tilde{b}_j \epsilon^{2k+1} j^{2k+1}, \\ e_{N+j} &= i \sum_{k=1}^{N-1} \tilde{a}_j \epsilon^{2k+1} j^{2k+1} \\ &\quad + \sum_{k=1}^{N-1} \tilde{b}_j \epsilon^{2k+1} j^{2k+1}, \\ 1 &\leq j \leq N, \end{aligned} \quad (5)$$

### 3 Deformations of the $P_9$ breather with sixteen parameters

We have constructed in [17, 16, 19, 18, 20, 22, 14] and in a series of articles on the archive hal, solutions for the cases from  $N = 1$  until  $N = 8$  with  $2N - 2$  parameters.

Here we make the study for the order nine. We don't give the analytic expression of the solution of NLS equation of order 9 with sixteen parameters

because of the length of the expression. For simplicity, we denote

$$\begin{aligned} d_3 &:= \det((n_{jk})_{j,k \in [1, 2N]}) \\ d_1 &:= \det((d_{jk})_{j,k \in [1, 2N]}). \end{aligned}$$

The number of terms of the polynomials of the numerator  $d_3$  and denominator  $d_1$  of the solutions are shown in the table below (Table 1) when other  $a_i$  and  $b_i$  are set to 0.

N=9	number of terms
$d_3(a_1, b_1, x, t)$	184554
$d_1(a_1, b_1, x, t)$	94332
$d_3(a_2, b_2, x, t)$	72174
$d_1(a_2, b_2, x, t)$	36894
$d_3(a_3, b_3, x, t)$	39813
$d_1(a_3, b_3, x, t)$	20347

Table 1: Number of terms for the polynomials  $d_3$  and  $d_1$  of the solutions of the NLS equation.

We construct figures to show deformations of the ninth Peregrine breather. The computations were done using the computer algebra systems Maple and TRIP [26]. For example, the computations of the determinants of  $Q(x, t)$  take only 3 hours for the figure 1 using the parallel kernel of the software TRIP on a workstation with 6 cores. We get different types of symmetries in the plots in the  $(x, t)$  plane. We give some examples of this fact in the following.

#### 3.1 Peregrine breather of order 9

If we choose  $\tilde{a}_i = \tilde{b}_i = 0$  for  $1 \leq i \leq 8$ , we obtain the classical Peregrine breather

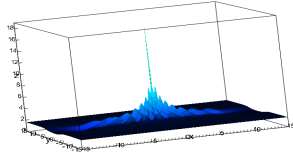


Figure 1: Solution to NLS,  $N=9$ , all parameters equal to 0, Peregrine breather  $P_9$ .

In the following, we give different deformations of this  $P_9$  breather depending on the choices of the parameters. The triangular patterns have already been explained for the orders until  $N = 7$  in [27]. The ring patterns, and the classification of the solutions to the NLS equation were already given in [28] using numerical methods until order  $N = 6$ . In that paper, the cases of order 7, 8, 9 and 10 given in table page 9 have been extrapolated. It was also pointed out that the number of peaks in the different figures for  $N = 9$  is  $N(N + 1)/2 = 45$ .

The results here are obtained from explicitly exact solutions.

### 3.2 Variation of parameters

With other choices of parameters, we obtain all types of configurations : triangles and multiple concentric rings configurations with a maximum of 45 peaks.

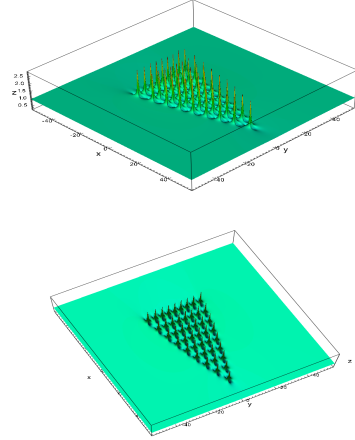


Figure 2: Solution to NLS,  $N=9$ ,  $\tilde{a}_1 = 10^3$  : triangle with 45 peaks; in bottom, sight of top.

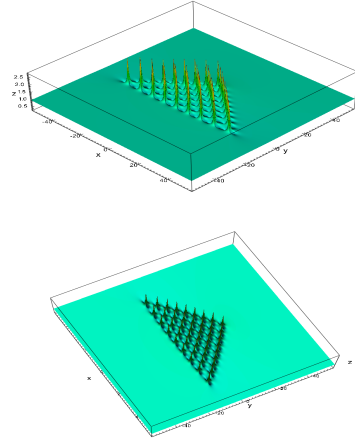


Figure 3: Solution to NLS,  $N=9$ ,  $\tilde{b}_1 = 10^3$  : triangle with 45 peaks; in bottom, sight of top.

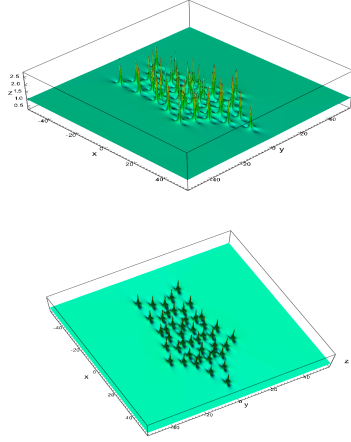


Figure 4: Solution to NLS,  $N=9$ ,  $\tilde{a}_2 = 10^5$  : 6 rings without a peak in the center; in bottom, sight of top.

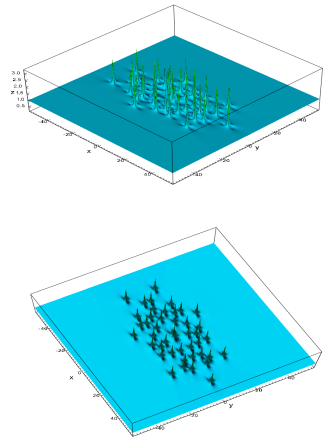


Figure 6: Solution to NLS,  $N=9$ ,  $\tilde{a}_3 = 10^7$  : 4 rings with in the center  $P_2$ ; in bottom, sight of top.

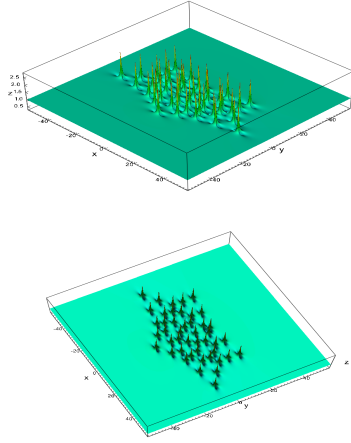


Figure 5: Solution to NLS,  $N=9$ ,  $\tilde{b}_2 = 10^5$  : 6 rings without a peak in the center; in bottom, sight of top.

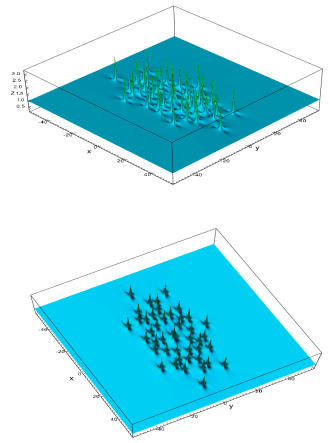


Figure 7: Solution to NLS,  $N=9$ ,  $\tilde{b}_3 = 10^7$  : 4 rings with in the center  $P_2$ ; in bottom, sight of top.

Similar figures were given in [28], and in all these figures, we have  $N(N+1)/2 = 45$  peaks as it was already extrapolated in the table page 9 of that article. In the same paper, it was al-

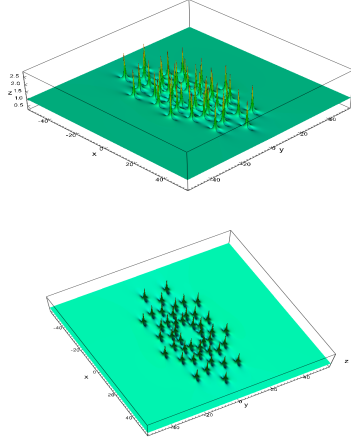


Figure 8: Solution to NLS,  $N=9$ ,  $\tilde{a}_4 = 10^9$  : 5 rings without a central peak; in bottom, sight of top.

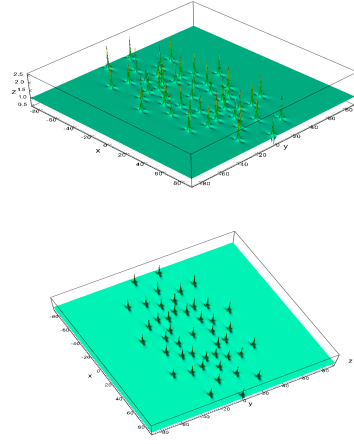


Figure 10: Solution to NLS,  $N=9$ ,  $\tilde{a}_5 = 10^{15}$  : 4 rings with in the center one peak; in bottom, sight of top.

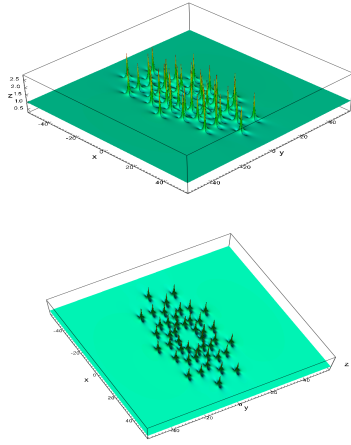


Figure 9: Solution to NLS,  $N=9$ ,  $\tilde{b}_4 = 10^9$  : 5 rings without a central peak; in bottom, sight of top.

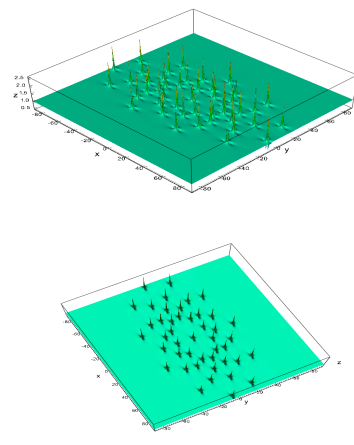


Figure 11: Solution to NLS,  $N=9$ ,  $\tilde{b}_5 = 10^{15}$  : 4 rings with in the center one peak; in bottom, sight of top.

ready conjectured that in the case of one ring, the ring has  $2N - 1 = 17$

peaks surrounding the  $P_{N-2} = P_7$  breather. It can be noted that when a more convenient normalization of the NLS equation is used, these rings can be transformed in circles.

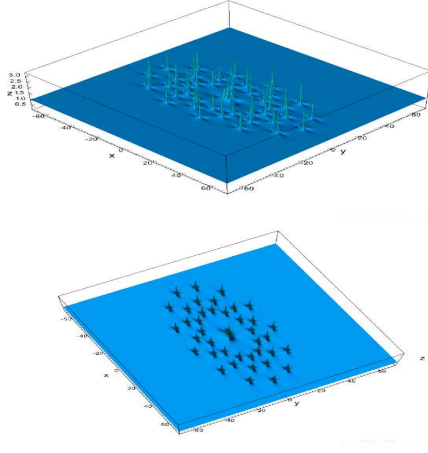


Figure 12: Solution to NLS,  $N=9$ ,  $\tilde{a}_6 = 10^{15}$  : 3 rings with in the center the Peregrine breather of order 3; in bottom, sight of top.

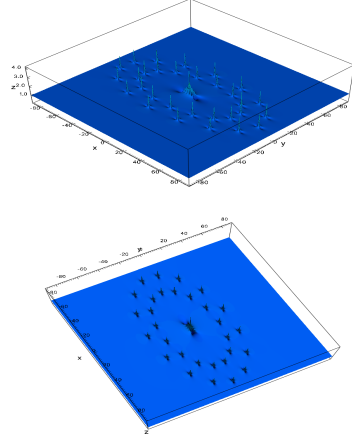


Figure 14: Solution to NLS,  $N=9$ ,  $\tilde{a}_7 = 10^{20}$  : two rings with in the center the Peregrine breather of order 5; in bottom, sight of top.

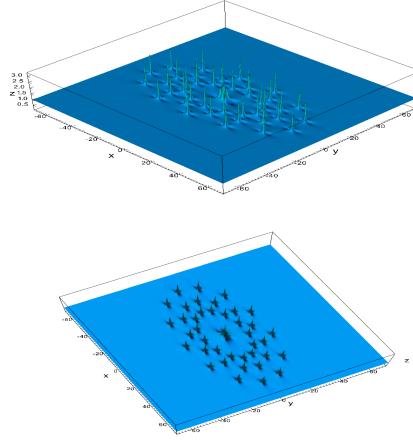


Figure 13: Solution to NLS,  $N=9$ ,  $\tilde{b}_6 = 10^{15}$  : 3 rings with in the center the Peregrine breather of order 3; in bottom, sight of top.

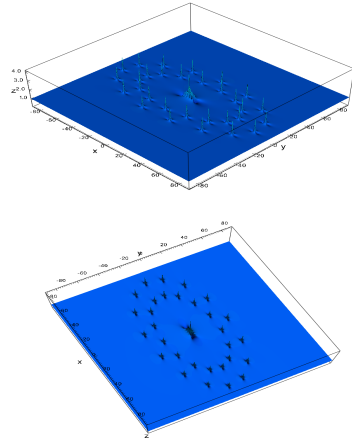


Figure 15: Solution to NLS,  $N=9$ ,  $\tilde{b}_7 = 10^{20}$  : two rings with in the center the Peregrine breather of order 5; in bottom, sight of top.

## 4 Conclusion

In the present paper we have constructed explicitly solutions to the NLS equa-

tion of order 9 with 16 real parameters. The explicit representation in terms of polynomials in  $x$  and  $t$  is too large to



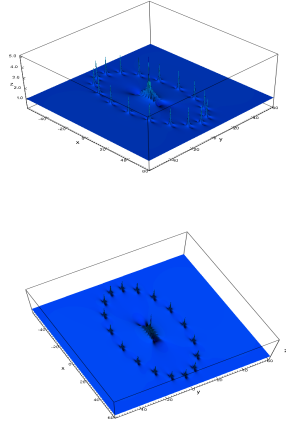


Figure 16: Solution to NLS,  $N=9$ ,  $\tilde{a}_8 = 10^{20}$  : one ring with in the center the Peregrine breather of order 7; in bottom, sight of top.

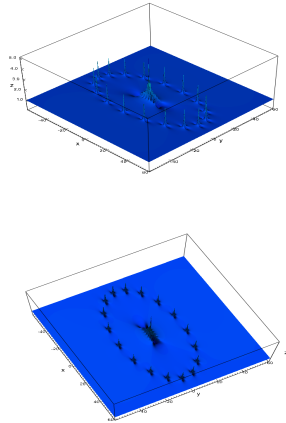


Figure 17: Solution to NLS,  $N=9$ ,  $\tilde{b}_8 = 10^{20}$  : one ring with in the center the Peregrine breather of order 7; in bottom, sight of top.

be published here.

In the case of the variation of one parameter, we obtain different types of

configurations with a maximum of 45 peaks.

By different choices of these parameters, we obtained new patterns in the  $(x; t)$  plane; we recognized ring shape as already observed in the case of deformations depending on two parameters [12, 10]. We get new triangular shapes and multiple concentric rings.

Many applications in nonlinear optics and hydrodynamics have been given recently : we can mention in particular the works of Chabchoub et al. [29] or Kibler et al. [30].

These explicit solutions of order 9 and their deformations with eighteen parameters are presented for the first time to our knowledge. These deformations of the ninth Peregrine breather give a better understanding of the phenomenon of appearance of rogue waves and their asymptotic behavior.

The study of the solutions to the NLS equation has been done until order  $N = 6$  by Akhmediev et al. in [28] and extrapolated until order  $N = 10$ . Present work gives explicitly exact solutions to the NLS equation at order 9; it verifies the conjectured classification given in [28].

It would be important to continue this study to try to classify solutions to the NLS equation in the general case of order  $N$  ( $N > 10$ ).

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